Lecture 10

Moment Generating and Characteristic functions

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The moment generating function

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- For RV X the moment generating function is

$$\theta(t) \equiv E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

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- Almost like the Laplace transform of $f_X(x)$!
  - Equivalent if we evaluate at $s = -t$.
  - Since Laplace xfm. inverses are known and unique (for given ROC), $\theta(t)$ contains the same information as $f_X(x)$.
- How is this used?

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  - Since Laplace xfm. inverses are known and unique (for given ROC), $\theta(t)$ contains the same information as $f_X(x)$.
- How is this used?
  - To compute moments simply, without integration!
  - To estimate $f_X(x)$ from sample moments.
  - Important analytical instrument: e.g. sums of RVs and proof of central limit theorem.

Computing the moments

- Use a series expansion

$$\theta(t) = E[e^{tX}] = E\left[1 + tX + \frac{(tX)^2}{2!} + \cdots + \frac{(tX)^k}{k!} + \cdots\right]$$

$$= 1 + t\mu + \frac{t^2}{2!}\xi_2 + \cdots + \frac{t^k}{k!}\xi_k + \cdots$$

- Now take the k^{th} derivative and evaluate at $t = 0$:

$$\frac{d^k}{dt^k}\theta(t)\bigg|_{t=0} = 0 + \cdots + 0 + \left(\frac{d^k}{dt^k}\frac{t^k}{k!}\right)\bigg|_{t=0}\xi_k + \left(\frac{d^k}{dt^k}\frac{t^{k+1}}{(k+1)!}\right)\bigg|_{t=0}\xi_{k+1}\cdots$$

$$= \xi_k + \frac{t}{1!}\bigg|_{t=0}\xi_{k+1} + \frac{t^2}{2!}\bigg|_{t=0}\xi_{k+2} + \cdots = \xi_k$$

Example: m.g.f. for $X \sim N(\mu, \sigma^2)$

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$$\theta(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x}{2\sigma^2}}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2(\mu+\sigma^2 t)x + \mu^2}{2\sigma^2}}\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{[x^2 - 2(\mu+\sigma^2 t)x + (\mu^2 + 2\mu\sigma^2 t + \sigma^4 t^2)] - 2\mu\sigma^2 t - \sigma^4 t^2}{2\sigma^2}}\, dx$$

Completing the square

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Completing the square

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}}\, e^{\mu t + \sigma^2 t^2/2}\, dx = e^{\mu t + \sigma^2 t^2/2}$$

Now compute 1st and 2nd moments!

Example: m.g.f. for $X \sim N(\mu, \sigma^2)$

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$$\theta(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2 - 2\mu x + \mu^2 - 2\sigma^2 t x}{2\sigma^2}} dx$$

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$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{[x-(\mu+\sigma^2 t)]^2}{2\sigma^2}} e^{\mu t + \sigma^2 t^2/2} dx = e^{\mu t + \sigma^2 t^2/2}$$

$$\xi_1 = \frac{d}{dt}\theta(t)\Big|_{t=0} = (\mu + \sigma^2 t)e^{\mu t + \sigma^2 t^2/2}\Big|_{t=0} = \mu,$$

$$\xi_2 = \frac{d^2}{dt^2}\theta(t)\Big|_{t=0} = \sigma^2 e^{\mu t + \sigma^2 t^2/2} + (\mu + 2\sigma^2 t)^2 e^{\mu t + \sigma^2 t^2/2}\Big|_{t=0} = \sigma^2 + \mu^2$$

Example: m.g.f. for binomial X

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- For discrete RV: $\theta(t) \equiv \sum_k e^{t x_k} P_X(x_k)$

- $\theta(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} = (pe^t + q)^n$

This is standard binomial series, of form $(a + b)^n$

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- $$\left.\frac{d}{dt}\theta(t)\right|_{t=0} = ?$$

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- $$\left.\frac{d}{dt}\theta(t)\right|_{t=0} = n(pe^t + q)^{n-1}(pe^t)\Big|_{t=0} = n(p+q)^{n-1}(p) = np = \mu$$

$$\left.\frac{d^2}{dt^2}\theta(t)\right|_{t=0} = ?$$

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$$\left.\frac{d^2}{dt^2}\theta(t)\right|_{t=0} = n(pe^t + q)^{n-1}(pe^t) + (pe^t)(n-1)n(pe^t + q)^{n-2}(pe^t)\Big|_{t=0}$$

$$= np + n(n-1)p^2 = (np)^2 + n(p - p^2) = (np)^2 + np(1-p) = \mu^2 + npq$$

σ^2

How would you estimate $f_X(x)$?

- Earlier we used a normalized histogram.
- Could use the moment generating function:
  - Estimate sample moments

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$$\hat{\xi}_r = \frac{1}{N}\sum_{i=1}^{N} x_i^r, \quad \text{where } x_i \text{ is } i^{\text{th}} \text{ sample drawn from } f_X(x)$$

  - Plug first n (as many as you have) into m.g.f. series expansion:

$$\hat{F}(s) = \mathcal{L}\left\{\hat{f}_X(x)\right\} = \hat{\theta}(-s) = 1 - s\hat{\mu} + \frac{s^2}{2!}\hat{\xi}_2 - \cdots + (-1)^n \frac{s^n}{n!}\hat{\xi}_n$$

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  - Compute inverse Laplace transform: $\hat{f}_X(x) = \mathcal{L}^{-1}\{\hat{F}(s)\}$ (tougher than it looks due to positive powers of s.)

Chernoff bound

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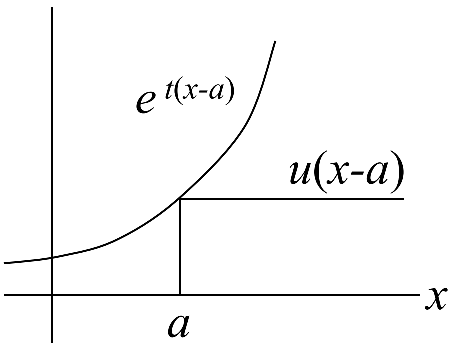
- $$P[X \geq a] \leq \min_{t>0} e^{-at}\theta_X(t)$$
- Proof:
  - Note $u(x-a) \leq e^{t(x-a)}$ for $t > 0$.



  - $$P[X \geq a] = \int_a^{\infty} f_X(x)\, dx = \int_{-\infty}^{\infty} f_X(x)\, u(x-a)\, dx$$
  - $$P[X \geq a] \leq \int_{-\infty}^{\infty} f_X(x)\, e^{t(x-a)}\, dx = e^{-at}\int_{-\infty}^{\infty} f_X(x)\, e^{tx}\, dx$$

$$\leq \; e^{-at}\theta_X(t), \quad \text{for } \; t > 0.$$

Note that $\theta_X(t) \geq 0$

Chernoff bound, observations

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- Discrete RV bound is the same form:

$$P[X \geq k] \leq \min_{t>0} e^{-kt} \theta_X(t)$$

- This is a tighter bound (though one-sided) than the Chebyshev inequality.
- Chernoff bound requires knowledge of the m.g.f. but may be easier to compute than

$$P[X \geq a] = \int_a^{\infty} f_X(x)\, dx$$

Example: Gaussian tail probability

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- Can we bound the probability that a random A.W.M. Dr. Jeffs meets is taller than him?
- $X \sim N(\mu = 69.7, \sigma^2 = 3.53^2)$, $a = 80$.
- $P[X \geq a] \leq \min_{t>0} e^{-at} \theta_X(t)$

(a slight exaggeration)

Example: Gaussian tail probability

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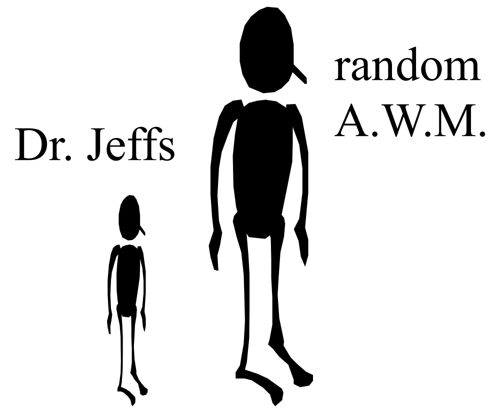

- Minimize the exponent:

$$\frac{d}{dt}\left(-(a-\mu)t + \sigma^2 t^2/2\right) = 0 \quad \rightarrow \quad \mu - a + \sigma^2 t = 0 \quad \rightarrow \quad t = \frac{a-\mu}{\sigma^2}$$

Note that since $t > 0$, we require $a > \mu$.

Example: Gaussian tail probability

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- Can we bound the probability that a random person Dr. Jeffs meets is taller than him?
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- $P[X \geq a] \leq e^{-(a-\mu)(a-\mu)/\sigma^2 + \sigma^2[(a-\mu)/\sigma^2]^2/2}$

$$\leq e^{-\frac{(a-\mu)^2}{2\sigma^2}} = e^{-\frac{(80-69.7)^2}{2(3.53)^2}} = 0.01417$$

$P[X \geq a] = 0.0018$ using erf( ). But now we have a closed form solution!

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Compare with Chebychev inequality:
$P[|X-\mu| \geq (a-\mu)] \leq \sigma^2/(a-\mu)^2 = 0.117$.
Assume symmetric tail probabilities and divide by 2: $P[X \geq a] \leq 0.0587$.
Chernoff is a much tighter bound!

The characteristic function

- For RV X the characteristic function is

$$\Phi_X(\omega) \equiv E[e^{j\omega X}] = \int_{-\infty}^{\infty} f_X(x)\, e^{j\omega x}\, dx$$

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- Almost like the Fourier transform of $f_X(x)$!
  - Equivalent if we evaluate at $\Omega = -\omega$.
  - Since Fourier xfm. inverses are known, it contains the same information as $f_X(x)$.
- How is this used?
  - Much like the moment generating function.
  - Computing moments simply.
  - Important analytical instrument: e.g. sums of RVs and proof of central limit theorem.

Sums of Independent RVs

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- Let $Z = X_1 + X_2 + \cdots X_N$, with all X_n mutually independent.
- $f_Z(z) = f_{X_1}(z) * f_{X_2}(z) * \cdots * f_{X_N}(z)$
- By the Fourier convolution theorem

$$\Phi_Z(\omega) = \Phi_{X_1}(\omega)\Phi_{X_2}(\omega) \times \cdots \times \Phi_{X_N}(\omega)$$

- So

$$f_Z(z) = \mathcal{F}^{-1}\{\Phi_{X_1}(\omega)\Phi_{X_2}(\omega) \times \cdots \times \Phi_{X_N}(\omega)\}$$

Computing moments with the characteristic function

- Much like with the moment generating function.
- By Taylor expansion:

$$\Phi_X(\omega) = E[e^{j\omega X}] = \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} \xi_n$$

- Take derivatives and set $\omega = 0$

$$\xi_n = \frac{1}{j^n} \frac{d^n}{d\omega^n} \Phi_X(\omega) \bigg|_{\omega=0}$$

Characteristic function for Gaussian

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- Let $X \sim N(\mu = 0, \sigma^2)$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} e^{j\omega x} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\{[x^2 - 2\sigma^2 j\omega x + (\sigma^2 j\omega)^2] - (\sigma^2 j\omega)^2\}} dx$$

$$= e^{-\frac{\sigma^2\omega^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}[x - (\sigma^2 j\omega)]^2} dx}_{1} = e^{-\frac{\sigma^2\omega^2}{2}}$$

- Sum of i.i.d. standard normals: Let $Z = X_1 + X_2 + \cdots X_N$
  - Can you find the pdf $f_Z(z)$ using the characteristic function?

Characteristic function for Gaussian

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- Let $X \sim N(\mu = 0, \sigma^2)$

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} e^{j\omega x} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\{[x^2 - 2\sigma^2 j\omega x + (\sigma^2 j\omega)^2] - (\sigma^2 j\omega)^2\}} dx$$

$$= e^{-\frac{\sigma^2\omega^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}[x - (\sigma^2 j\omega)]^2} dx = e^{-\frac{\sigma^2\omega^2}{2}}$$

(the integral $\rightarrow 1$)

- Sum of i.i.d. standard normals: Let $Z = X_1 + X_2 + \cdots X_N$

$$\Phi_Z(\omega) = \Phi_{X_1}(\omega)\Phi_{X_2}(\omega) \times \cdots \times \Phi_{X_N}(\omega) = \Phi_X^N(\omega)$$

$$= \left(e^{-\frac{\omega^2}{2}} \right)^N = e^{-\frac{N\omega^2}{2}} \longrightarrow Z \sim N(\mu = 0, \sigma^2 = N)$$

Sums of binomial RVs (the power of char. fun. analysis)

- Let $Z = X+Y$, X and $Y \sim$ i.i.d. $P_X(k) = P_Y(k) = \binom{n}{k} p^k q^{n-k}$
- Find $P_Z(k)$ using convolution:
- OK, but it is hard to interpret this form.
  Try characteristic function.

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- Find $P_Z(k)$ using convolution:

$$P[Z = k] = P_X(k) * P_Y(k) = \sum_i P_X(i) P_Y(k-i)$$

$$= \sum_i \binom{n}{i} p^i q^{n-i} \binom{n}{k-i} p^{k-i} q^{n-(k-i)} = p^k q^{2n-k} \sum_i \binom{n}{i}\binom{n}{k-i}, \quad 0 \le k \le 2n$$

- OK, but it is hard to interpret this form.
Try characteristic function.

Sums of binomial RVs (the power of char. fun. analysis)

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- Compute characteristic function

$$\Phi_X(\omega) = \Phi_Y(\omega) = \sum_{k=0}^{n} e^{j\omega k} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^{j\omega})^k q^{n-k} = (pe^{j\omega} + q)^n$$

so:

$$\Phi_Z(\omega) = E[e^{j\omega(X+Y)}] = E[e^{j\omega X}]E[e^{j\omega Y}] = (pe^{j\omega} + q)^{2n}$$

Due to independence

- We recognize this as the characteristic function for binomial with parameters $2n$ and p, so

$$P_Z(k) = \binom{2n}{k} p^k q^{2n-k}$$

This is much more informative than the previous form. *We now clearly see sums of independent binomials are binomial.*

Sum of independent uniforms

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- Let $Z = X+Y$, X and $Y \sim$ i.i.d. U(-a/2,$a/2$).
- $f_X(x) = f_Y(x) = \frac{1}{a}\,\mathrm{rect}(x/a)$
- $\Phi_X(\omega) = ?$

Sum of independent uniforms

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- $$f_X(x) = f_Y(x) = \frac{1}{a}\,\mathrm{rect}(x/a)$$
- $$\Phi_X(\omega) = \frac{\sin(a\omega/2)}{a\omega/2}$$

From table in your 380 text!

Sum of independent uniforms

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- Let $Z = X+Y$, X and $Y \sim$ i.i.d. $U(\text{-a}/2, a/2)$.
- $$f_X(x) = f_Y(x) = \frac{1}{a}\,\mathrm{rect}(x/a)$$
- $$\Phi_X(\omega) = \frac{\sin(a\omega/2)}{a\omega/2}$$ ← From table in your 380 text!
- $$\Phi_Z(\omega) = \left(\frac{\sin(a\omega/2)}{a\omega/2}\right)^2$$ ← i.d. RV sums → pdf convolution → char. fun. product.

Sum of independent uniforms

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- $$\Phi_Z(\omega) = \left(\frac{\sin(a\omega/2)}{a\omega/2}\right)^2$$ ← i.d. RV sums → pdf convolution → char. fun. product.
- $$f_Z(z) = \frac{1}{a}\left(1 - \tfrac{|z|}{a}\right)\mathrm{rect}\left(\tfrac{z}{2a}\right)$$ ← From inverse transform table in your 380 text!

Sum of independent uniforms

- Let $Z = X+Y$, X and $Y \sim$ i.i.d. $U(-a/2,a/2)$.
- $f_X(x) = f_Y(x) = \dfrac{1}{a}\,\mathrm{rect}(x/a)$
- $\Phi_X(\omega) = \dfrac{\sin(a\omega/2)}{a\omega/2}$
- $\Phi_Z(\omega) = \left(\dfrac{\sin(a\omega/2)}{a\omega/2}\right)^2$
- $f_Z(z) = \dfrac{1}{a}\left(1-\frac{|z|}{a}\right)\mathrm{rect}\left(\frac{z}{2a}\right)$

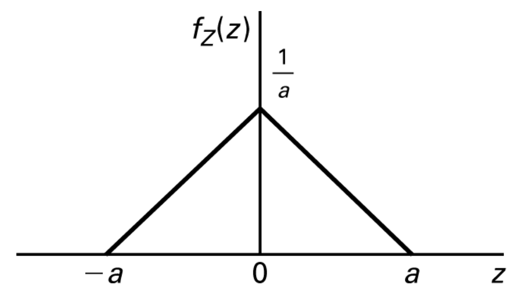

Figure 4.7-1
The pdf of $Z = X + Y$ when X and Y are i.i.d. uniformly distributed in $(-a/2, a/2)$.

Fair games and expectation

- In a "fair" betting game, the expected return equals the amount of the bet.
- Lottery: 40% goes to "education fund," 10% to "administration."
  - Is it a fair game?

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- Lottery: 40% goes to "education fund," 10% to "administration."
  - Is it a fair game?
  - $E[X] = 0.5$
- Not fair!
  Moral: don't gamble!

Roulette

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- Black and red nos. 1-36 , and 2 green zeros.
- House wins on zero.
- Can bet single number, (returns $36 for $1 bet), or color ($2 for $1 bet).
- Is it a fair game?

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- House wins on zero.
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- Is it a fair game?
  - Bet on single number:

$$E[X] = (36)(1/38) + 0(1/38) + \cdots + 0(1/38) = 0.947$$

  - Bet on color:

$$E[X] = (2)(18/38) + 0(2/38) + 0(18/38) = 0.947$$

- Not fair! Moral: Don't Bet!

The run of heads game (A.K.A. the St. Petersburg problem)

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- Would you pay $20 to play this game?
  - Toss a fair coin until tails comes up.
  - If tails appears on first flip you get $1.
  - If it appears on second flip you get $2.
  - Winnings double with each flip until a tail appears.
- What is the expected return?
- What is a "fair" price to pay?
- What is a reasonable price?

The run of heads game (A.K.A. the St. Petersburg problem)

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- First posed in 1713 by Nicholas Bernoulli
  - Published in annals of St. Petersburg Academy in 1738.
  - Became a *cause celebre*, a paradox showing how expectation and notion of a fair game could produce an unreasonable result.
- What is the expected return?
- What bet makes this a fair game?
- What is a reasonable price?

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$$E[X] = \sum_{i=0}^{\infty} x_i P(x_i) = 1 \times \tfrac{1}{2} + 2 \times \tfrac{1}{4} + \cdots \frac{2^i}{2^{i+1}} + \cdots = \sum_{i=0}^{\infty} \tfrac{1}{2} = \infty$$

- What bet makes this a fair game?
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  $\$\infty$

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- What bet makes this a fair game?

  $∞

- What is a reasonable price?

  Common sense says about $3.

The run of heads game (A.K.A. the St. Petersburg problem)

- What is wrong here?
  - Cousin Daniel Bernoulli proposed another solution.
  - Claimed players here act more out of prudence than fairness.
  - Proposed replacing “mathematical” expectation with “moral” expectation.

The run of heads game (A.K.A. the St. Petersburg problem)

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- What is wrong here?
  - Cousin Daniel Bernoulli proposed another solution.
  - Claimed players here act more out of prudence than fairness.
  - Proposed replacing “mathematical” expectation with “moral” expectation.
- This became the basis for a new theory which included risk (a property associated with the player) along with probability.
  - An incremental loss of $10 means more to a poor person than to a rich person. The poor person makes a smaller bet.
  - This suggests risk (or expected gain) should scale logarithmically:

x = bet amount, Δrisk = incremental loss pain per additional dollar bet.

$$\text{Say that } \Delta\text{risk} \propto 1/x. \quad \to \quad dr \propto 1/x \quad \to \quad r \propto \ln x + c$$

The run of heads game (A.K.A. the St. Petersburg problem)

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- Solve the problem using "moral expectation" or log risk:

$$E[R] = E[\log_2 X] = \sum_{i=0}^{\infty} \log_2 x_i P(x_i) = \sum_{i=0}^{\infty} \log_2 (2^i)/2^{i+1} = \sum_{i=0}^{\infty} i\left(\tfrac{1}{2}\right)^{i+1} = \sum_{i=1}^{\infty} i\left(\tfrac{1}{2}\right)^{i+1}$$

$$\text{Recall } \sum_{i=0}^{\infty} a^i = \frac{1}{1-a}, \text{ so } \frac{d}{da}\sum_{i=0}^{\infty} a^i = \frac{d}{da}\frac{1}{1-a} \rightarrow \sum_{i=1}^{\infty} i a^{i-1} = \frac{1}{(1-a)^2}$$

$$\text{Thus } E[R] = \tfrac{1}{4}\sum_{i=1}^{\infty} i\left(\tfrac{1}{2}\right)^{i-1} = \frac{1}{4(1-1/2)^2} = 1.$$

- Now this is a "reasonable" result!
  - Note that choice of log base is arbitrary, just scales $E[R]$.

Pascal’s wager

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- Ca. 1669 Blaise Pascal posed a mathematical argument to convert “sporting libertines” to a life of religious devotion.
  - This is the first record of expectation as a product of probability and the return.

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  - Suppose an unbeliever ascribes small probability, ε, to the existence of God.
  - If God exists, the return, X, for gambling a finite life of devotion is eternal, infinite bliss.
- Should a reasonable man take the wager?

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- Should a reasonable man take the wager?
  - $E[X] = \infty\varepsilon + 0(1-\varepsilon) = \infty.$
  - $E[X] >>$ finite cost of pious life, so YES, accept the wager, live virtuously!